The Role of Porosity in Filtration VII Effect of Side-Wall Friction in Compression-Permeability Cells

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The effect of side-wall friction on the uniformity of packing of beds in compression-permeability cells is examined. A large portion of pressure applied to the top of a compressible bad is absorbed in wall friction, resulting in a nonuniformly packed bed. A simplified analysis of wall friction and its effect on porosity and permeability is presented.

All experimenters involved in flow through porous media are urged to examine the effects of stress distribution on the structure and uniformity of the packing. In general, compressible beds of small particles will be strongly affected by the walls.

Design of new filter equipment and the analysis of commercial filtration operations require knowledge of overall filtration resistance coefficients. Calculation of overall coefficients depends upon integration of point or local values over the thickness of the cake. Local coefficients may be obtained directly through use of a compression-permeability (C-P) cell (Figure 1) (*1, 2, 3, 6, 8*) or indirectly by calculations based upon data relating the variation of overall resistance to pressure drop (*14*).

COMPRESSION-PERMEABILITY CELL

The C-P cell is a device for obtaining local values of porosity and filtration resistance coefficients as a function of applied compressive pressure. Basically the cell consists of a vertical cylinder with a movable piston through which a mechanical load is applied at the top of a confined bed of solids. Permeabilities (or filtration resistance coefficients) are obtained by permitting liquid to percolate through the cake under a low hydrostatic head. Both permeability and porosity (fraction of void space) are obtained as functions of the applied mechanical pressure. It is generally assumed that the mechanical pressure produces the same effects as the cumulative frictional drag of liquid passing through the cake. When the frictional drag divided by the cross-sectional area in a filter cake equals the mechanically applied pressure in the C-P cell, the porosity and permeability in the filter cake and C-P cell are assumed to be identical. Although that assumption is fundamental to present filtration theory, it is only an approximation which requires careful analysis and study in practice.

NONUNIFORMITY OF CAKE

As originally developed by Ruth (*6*) and used by other investigators (*1, 3, 8*) the bottom of the C-P cell, on which the cake rested, was an integral part of the side walls while the top consisted of a movable piston. In the field of soil mechanics (*10, 11, 17*), consolidometers (devices similar to C-P cells in which only porosity is measured) had been developed in which it was possible to measure the transmitted as well as the applied load. In 1960, Haynes (*2*) built a new type C-P cell utilizing freely floating pistons on both the bottom and top of the cake (Figure 1). Haynes' device permitted measurement of wall friction which was found to have a significant effect on the uniformity of the cake in the C-P cell.

Few investigators have recognized the importance of side-wall friction in filtration theory. Usual practice involves tubes, cups, or funnels of small diameter in the determination of resistance data. It is highly doubtful that extrapolation from small diameter to commercial size equipment could be considered as being reliable. The absence of properly instrumented filter presses in the chemical industry has resulted in a paucity of data for purposes of correlation. Grace (*1*) reported data which show that the porosity of a thick cake tested in a C-P cell was greater at the bottom than at the top. Side-wall friction was undoubtedly responsible for the nonuniformity of porosity. Taylor (*11*) and Welch (*17*) recognized that side-wall friction was present in a consolidometer and measured the ratio of transmitted to applied pressure. However, those authors did not consider a correction necessary when the results were utilized in the practical field of soil mechanics. Shirato et al. (*9*) and his student Sawamoto (*7*) developed cells based on Haynes' apparatus and carried on an extensive analysis of side-wall friction.

If the compressive solids pressure were uniform throughout the cake in a C-P cell, then the experimental data could be used as point values in integrations to give overall resistances in actual filtration processes. For the simplified case in which flow rate through the cake is considered constant (generally valid for dilute slurries in which the volume fraction of solids is small compared to the volume fraction of solids in the surface layer of the cake), the overall filtration resistance α_R as defined by Ruth (*6*) may be written as

$$\alpha_R = \frac{\Delta p_c}{\int_0^{\Delta p_c} \frac{dp_s}{\alpha_x}} \tag{1}$$

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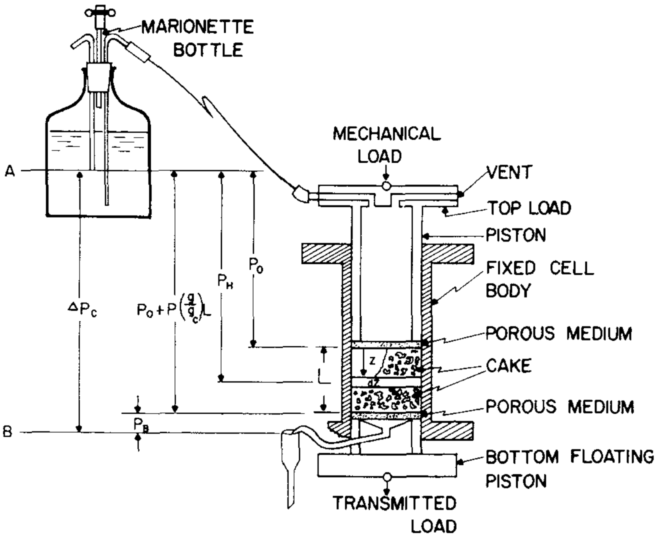

Fig. 1. Schematic diagram of compression-permeability cell.

Unfortunately, an appreciable frictional effect between the wall of the C-P cell and the cake distorts the stress distribution and results in nonuniformity of an undetermined magnitude. Average values obtained from C-P cells should be corrected before being used for scale-up purposes.

Grace (*1*) and Shirato et al. (*8*) have reported good correlation between resistances calculated by means of Equation (1) utilizing values of α_x obtained from C-P cells and values of α_R resulting from constant pressure filtrations. Shirato also used modifications of Equation (1) involving variable internal flow rate as required for concentrated slurries. Both authors used filters with dimensions comparable to their C-P cells; consequently, they had similar cake distortions in each case.

In this paper, a C-P cell with floating pistons is discussed, and a simplified treatment of wall friction is presented. Correction factors are developed for porosity and filtration resistance. The crude assumptions made in this paper lead to simple formulas which should be viewed only as rough approximations. In other articles in this series, a further analysis of the characteristics of C-P cells will be presented.

COMPRESSIVE PRESSURE AND FILTRATION DRAG

Filter cakes are compressed by accumulative frictional drag on individual particles. As liquid flows through a filter bed, hydraulic pressure p_x decreases and the total drag F_s increases. For beds with particles in point contact and in which momentum change can be neglected, the drag and hydraulic pressure are related by (*8, 15*)

$$dF_s + A dp_x = 0 \tag{2}$$

where A is the total cross-sectional area. The force F_s results from frictional drag and is transmitted at points of contact between particles. The compressive pressure (termed effective pressure in soil mechanics) is defined by $dp_s = dF_s/A$ reducing Equation (2) to

$$dp_s + dp_x = 0 \tag{3}$$

Integrating (3) at a given time yields

$$p_s(x, t) + p_x(x, t) = p(t) \tag{4}$$

where the total applied hydraulic pressure p can vary with time; and p_s and p_x vary with both time and position. Only in constant pressure filtration would $p(t)$ be constant.

Consider a filter and a corresponding C-P cell in which there are unidirectional liquid flows and the same cross-sectional area. A mechanical load in the C-P cell produces compression while the frictional drag compresses the filter cake. In the absence of wall effects, the mechanical pressure would be uniformly transmitted throughout the bed leading to a uniform bed structure. In a filter cake, the drag is accumulative; and the compressive pressure varies throughout the cake. At that point in the cake where the compressive pressure ($p_s = p - p_x$) equals the applied mechanical pressure in the C-P cell, it is assumed that the local porosity ϵ_x and the local filtration resistance α_x in the cake are equivalent to the values obtained in the C-P cell. Based upon those assumptions, investigators have employed C-P cell data for predicting porosity and hydraulic pressure variations in compressible cakes.

WALL FRICTION

When a particulate bed is in contact with a solid surface, both static and dynamic friction are present. In general, such friction plays an important part in many types of physical processes. As an example, free-floating pilings depend upon static friction between soil and piling for their effectiveness.

In a filter unit, wall friction alters the fundamental expressions given in Equations (2) and (3). In general, the compressive solids pressure would vary in a two- or three-dimensional sense. However, if it is assumed that the pressure is uniform at given cake depths, the solids pressure would be given by

$$p_s = p - p_x - p(\text{wall}) \tag{5}$$

where $p(\text{wall})$ represents the wall friction F_w divided by the cross-sectional area. The force F_w is perpendicular to the area used to compute the frictional wall pressure.

In a C-P cell, conditions are somewhat different from those in a filter. The total load at a distance, z, from the top (Figure 1) is affected by the mechanical load, the weight of the buoyed solids, hydrostatic pressure, frictional drag due to flow, and the wall friction; the total pressure is given by

$$p(\text{total}) = \text{net mech. load} + \text{wt. of buoyed solids} + \text{hydraulic pressure} + \text{drag} - \text{wall friction} \tag{6a}$$

$$= p_m + (\rho_s - \rho)(1 - \epsilon_{avz})\left(\frac{g}{g_c}\right)z + p_x + (p_H - p_x) - p_w \tag{6b}$$

where ϵ_{avz} is the average porosity in the distance 0 to z. The net mechanical load consists of the buoyed weight of the top piston plus any external force. It should be noted that p_x is the actual hydraulic pressure at height z. The drag can be calculated from Equation (4). It equals the local hydrostatic pressure minus the local hydraulic pressure. If p_0 is the pressure at the surface of the cake due to the Marionnette head tank, then the hydrostatic pressure at height z is

$$p_H = p_0 + \rho\left(\frac{g}{g_c}\right)z = p_0 + \gamma z \tag{7}$$

where γ is the specific weight of the liquid. The total pressure drop across the cake equals Δp_c as shown in Figure 1.

In Figure 2, the relationship of the various pressures as a function of height is shown. The line ACDE represents the hydrostatic pressure line starting at level A in Figure 1 and terminating at level B. From A-C in Figure 2, the pressure p_0 above the cake is developed. From C-D the hydrostatic pressure increases through the cake; and D-E represents the suction in the drainage below the cell.

The hydraulic pressure in the cell is p_0 at point C where $z = 0$. If the solids in the cell were uniform, there would be a straight line pressure drop as shown by the dotted line from C to $z = L$ in Figure 2. The actual hydraulic pressure distribution will be given by the solid line marked p_x. The slope will be greatest at the top of the cell ($z = 0$), because the compressive solids pressure will be at its maximum at that point. As friction builds up and reduces the compressive solids pressure, the filtration resistance decreases; and the slope of the hydraulic pressure curve will also decrease. The hydraulic pressure reaches a pressure equal to ($-p_B$) at the bottom of the cake.

The effective compressive pressure (*8, 12*) producing compression (p_z or σ_z in a C-P cell and p_s in a filter cake) is given by the total pressure minus the hydraulic pressure. Utilizing Equations (6*b*) and (7) with $p_z = p(\text{total}) - p_x$ leads to

$$p_z = p_m + [\rho_s(1 - \epsilon_{avz}) + \rho\epsilon_{avz}]\left(\frac{g}{g_c}\right) z + p_0 - p_x - p_w \quad (8)$$

Equation (8) can be rewritten as

$$p_z = p_m + p_0 + \gamma z - p_x + (1 - \epsilon_{avz})(\gamma_s - \gamma) z - p_w \quad (9)$$

The term ($p_0 + \gamma z - p_x$) is the frictional drag, and $(1 - \epsilon_{avz})(\gamma_s - \gamma)z$ is the buoyed weight of the solids.

At $z = 0$, $p_x = p_0$, $p_w = 0$; and Equation (9) reduces to $p_z = p_m$. At the bottom of the cake $z = L$ and $p_x = -p_B$ (note the hydraulic pressure variation in Figure 2 at the exit B); and the total compressive pressure is given by

$$p_{z=L} = p_m + \gamma_{av}L + p_0 + p_B - p_w \quad (10)$$

where $\gamma_{av} = \gamma_s(1 - \epsilon_{av}) + \gamma\epsilon_{av}$ is the average specific weight for the entire cake. Equation (9) can be rewritten in the following form

$$p_{z=L} = p_m + p_0 + p_B + \gamma L + (\gamma_s - \gamma)(1 - \epsilon_{av}) L - p_w \quad (10a)$$

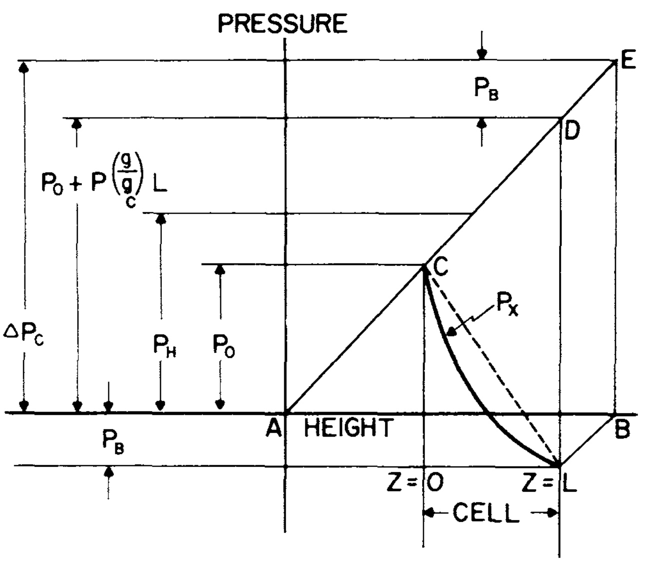

Fig. 2. Hydraulic pressure relations in compression-permeability cell.

The term ($p_0 + p_B + \gamma L$) is the total frictional pressure drop and equals Δp_c. The second to last term is the buoyed weight of the solid bed. The compressive pressure at the bottom of the bed is then given by

$$p_{z=L} = \text{mech. pressure} + \text{drag} + \text{buoyed solids-wall friction} \quad (10b)$$

The magnitudes of the terms in Equations (10*a*) and (10*b*) can vary quite widely. The relative values may be compared by assuming reasonable ranges for the parameters as follows.

	Low	High
Porosity	0.4	0.9
Solid, sp.gr.	0.8	3.0
Cake thickness, in.	0.25	4
Hydraulic head, in. water	3	48

Then the terms in Equations (10) and (11) have the following approximate ranges expressed in lb./sq.in.

	Minimum	Maximum
p_m	0	100
p_0	0.1	1.7
$(1 - \epsilon_{av})(\gamma_s - \gamma)L$	−0.02	0.2
γL	0.008	0.15

The magnitude of p_w may range from 10-70% of p_z.

In the past, investigators (*1, 3, 6, 9*) have assumed that $p_z = p_m$. Lu (*4*) thoroughly investigated the effects of each term in Equation (11) and conclusively showed that they are important in the low-pressure range. Experimentally, porosity and permeabilities can only be obtained with great difficulty at very low pressures. Yet, in theoretical formulas, the limiting values, ϵ_i and α_i, which are approached as the compressive pressure approaches zero are highly important and exert a substantial effect on numerical calculations.

EMPIRICAL FORMULAS FOR RESISTANCE AND POROSITY

While numerical methods must be used with highly compressible materials, there are many cases where a power function can be conveniently used to relate α and ϵ to the compressive pressure p_s. As p_s approaches 0, the resistance and porosity approach limiting values of α_i and ϵ_i. Analytically it is possible to represent data as follows:

$$\epsilon(\text{local}) = \epsilon_x = \epsilon_0 p_s^{-\lambda} \qquad p_s \geq p_i \quad (12)$$

$$= \epsilon_0 p_i^{-\lambda} = \epsilon_i \qquad p_s \leq p_i \quad (13)$$

$$\alpha(\text{local}) = \alpha_x = \alpha_0 p_s^{n} \qquad p_s \geq p_i \quad (14)$$

$$= \alpha_0 p_i^{n} = \alpha_i \qquad p_s \leq p_i \quad (15)$$

Equations (12) to (15) state that ϵ_x and α_x follow power functions to a low pressure p_i (generally ranging from 0.1 to 5 lb./sq. in.) below which they are constant. While other approximations can be used, Equations (12) to (15) lend themselves to mathematical manipulation (*4, 13 to 15*).

In Figure 3, typical logarithmic plots are illustrated for a mixture consisting of 25% kaolin and 75% supercel. Experimentally, it is exceedingly trying to obtain accurate values at very low pressures; consequently, values of p_i, ϵ_i, and α_i are seldom more than approximations. Better experimental methods need to be developed for obtaining low pressure data.

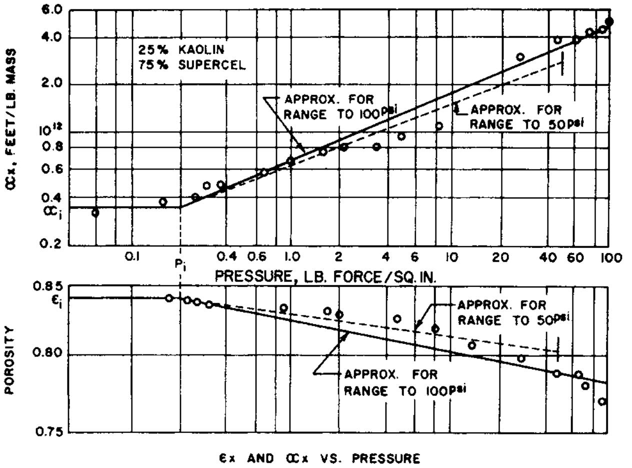

Fig. 3. Logarithmic plots of porosity and filtration resistance versus compressive pressure.

Depending upon the range of interest, various approximations may be used for representing the data. Normally, it is not possible to stretch the approximation beyond 100 lb./sq. in. While better data could be shown, Figure 3 is typical of what is frequently encountered. In the 100 lb./sq. in. range, the data can be approximated with the following parameters:

$$\epsilon_x = 0.827\,\psi_x^{-0.12} \qquad x \geq 0.2 \text{ lb./sq. in.} \qquad (16)$$

$$\epsilon_i = 0.84 \qquad x \leq 0.2 \text{ lb./sq. in.} \qquad (17)$$

$$\alpha_x = 7(10'')\,\psi_x^{0.428} \qquad x \geq 0.2 \text{ lb./sq. in.} \qquad (18)$$

$$\alpha_i = 3.5(10'') \qquad x \leq 0.2 \text{ lb./sq. in.} \qquad (19)$$

where ψ_x is the pressure in lb.$_f$/sq. in. In general, both the porosity and filtration resistance can be represented by the power functions for moderately compressible materials. When n approaches 0.5-0.7, the approximations usually fall down. For highly compressible materials like solka-floc (high permeability) or polystyrene latex (low permeability), Equations (12) to (15) are not satisfactory.

SIMPLIFIED WALL FRICTION THEORY

In order to evaluate the effect of wall friction in a C-P cell, it is necessary to estimate the solids compressive pressure at various depths. While a two- or three-dimensional partial differential equation would be required for an exact solution, an approximate analysis (*10*) is possible if certain simplifying assumptions are made. No accurate analytical methods have been developed for predicting stress distribution in particulate beds of remolded soils or filter cakes.

Basic assumptions necessary to the solution of the problem are: (1) the local lateral pressure or stress σ_r is proportional to the local vertical pressure or stress σ_z, (2) the vertical pressure is uniform at constant depth, and (3) the coefficient of friction between cake and cell is constant. The first assumption can be expressed in the form

$$\sigma_r = K_0\,\sigma_z \qquad K_0 \leq 1 \qquad (20)$$

where K_0, the coefficient of lateral earth pressure, is a physical characteristic of a given material. Sowers and Sowers (*10*) reported values of K_0 ranging from 1.0 (Poisson ratio equal to 0.5) for soft clays to 0.4 for dense sand or gravel.

The frictional shear force at the wall is given by

$$d\tau_{rz} = f\sigma_r \pi D dz = K_0 f \sigma_z \pi D dz \qquad (21)$$

Shirato et al. (*8*) presented Equation (21) with an added term representing cohesion between the wall and the cake. The term $fK_0\sigma_z$ was replaced by $fK_0\sigma_z + C$. While cohesive forces are commonly included in problems involving clays, Shirato considered the term as acting only at the wall and not within the cake. He also omitted drag and gravitational forces and did not include a term for cohesion within the cake itself. In view of the highly approximate nature of the assumptions, it is doubtful that cohesion, which at best would be small, warrants inclusion.

The frictional wall pressure dp_w is given by

$$dp_w = \frac{d\tau_{rz}}{\pi D^2/4} = \frac{4fK_0}{D}\,\sigma_z dz \qquad (22)$$

The vertical unit stress σ_z can be placed equal to the effective vertical compressive pressure p_z. The differential of p_z can be obtained from Equation (8), thus

$$dp_z = \gamma_x dz - dp_x - dp_w \qquad (23)$$

where $\gamma_x = \gamma_s\,(1 - \epsilon_x) + \gamma\epsilon_x$. The hydraulic pressure drop dp_x is given by

$$dp_x = \alpha_x \frac{\rho_s}{g_c}\,(1 - \epsilon_x)\mu\, q dz = \alpha_x \frac{\gamma_s}{g}\,(1 - \epsilon_x)\mu\, q dz \qquad (24)$$

On substituting (22) and (24) in (23), one obtains

$$\frac{dp_z}{dz} + \frac{4K_0 f}{D}\,p_z = \gamma_x - \alpha_x \frac{\gamma_s}{g}\,(1 - \epsilon_x)\,\mu q \qquad (25)$$

The porosity does not vary greatly, and ϵ_x and γ_x can be replaced by their average values ϵ_{av} and γ_{av}. The filtration resistance may have a greater variation and cannot in general be assumed constant. Replacing α by values given in Equations (13) and (14) yields

$$\frac{dp_z}{dz} + \frac{4K_0 f}{D}\,p_z = \gamma_{av} - \alpha_i \frac{\gamma_s}{g}\,(1 - \epsilon_{av})\mu\, q = B \qquad (26)$$

$$p_z \leq p_i$$

and

$$\frac{dp_z}{dz} + \frac{4K_0 f}{D}\,p_z = \gamma_{av} - \alpha_0 p_z^{\,n} \frac{\gamma_s}{g}\,(1 - \epsilon_{av})\mu\, g \qquad (27)$$

$$p_z \geq p_i$$

If Shirato's (*8*) cohesion were included, it would become an additive constant in Equations (26) and (27). In Equation (26) the last term represents the hydraulic pressure drop per unit of bed depth and may be written as $\Delta p_c/L$ where Δp_c is the total pressure drop across the solid bed. For static tests without flow, $q = 0$; and the hydraulic pressure drop term in Equations (26) and (27) would be eliminated.

Equation (26) has a simple solution of the form

$$p_z = \left(p_m - \frac{BD}{4fK_0}\right) e^{-4fK_0 z/D} + \frac{BD}{4fK_0} \qquad (28)$$

The solution of the nonlinear Equation (27) is more involved. The magnitude of each term in Equation (27) needs to be studied for a wide variety of conditions. Taking a typical situation with $4K_0 f = 0.5$, (relatively small value), $D = 2$ in., $L = 1.0$ in., a head of 2 ft. of water, $\gamma_{av} = 100$ lb.$_f$/cu. ft., $p_m = 10$ lb./sq. in., and assuming 70% of the applied load transmitted to the bottom piston, the terms in Equation (27) may be compared as follows:

Term	Magnitude, (lb.$_f$/(sq. ft.) (ft.)
$4K_0 f\ p_z/D$	17,000 at top
	12,000 at bottom
γ_{av}	100
$\Delta p_c/L$	1,500

A 1-in. cake is really too thick; and for thinner beds, the weight of the bed becomes small in comparson to other factors. The authors believe that very low L/D ratios should be preserved, preferably less than 0.1. When L/D is less than 0.1, the term γ_{av} in Equation (27) can be omitted if normal friction is involved.

If γ_{av} is eliminated from Equation (27), it becomes a Bernoulli differential equation, and its solution subject to $p_z = p_m$ at $z = 0$ is

$$p_z^{1-n} = \left[p_m^{1-n} + \frac{D}{4K_0 f} \alpha_0 \frac{\gamma_s}{g} (1 - \epsilon_{av}) \mu q \right] e^{-4K_0 f(1-n)z/D} - \frac{D}{4K_0 f} \alpha_0 \frac{\gamma_s}{g} (1 - \epsilon_{av}) \mu q \quad (29)$$

If a static test is run with $q = 0$ and the pressures are high enough to neglect the weight of the cake, Equation (29) reduces to

$$p_z = p_m \, e^{-4K_0 f z/D} \quad (30)$$

Experimental procedures in the C-P cell are carried out in a manner such that the average filtration resistance is calculated from the equation

$$\frac{\Delta p_c}{L} = \alpha_{av} \frac{\rho_s}{g_c} (1 - \epsilon_{av}) \mu q = \alpha_{av} \gamma_g (1 - \epsilon_{av}) \mu q \quad (31)$$

Substituting the value of α_{av} from (31) in (29) and obtaining the ratio of transmitted pressure p_T (for $z = L$) to p_m yields

$$\left(\frac{p_T}{p_m} \right)^{1-n} = \left[1 + \left(\frac{\alpha_0 p_m^n}{\alpha_{av}} \right) \left(\frac{D}{4K_0 f L} \right) \left(\frac{\Delta p_c}{p_m} \right) \right] e^{-4K_0 f(1-n)L/D} - \left(\frac{\alpha_0 p_m^n}{\alpha_{av}} \right) \left(\frac{D}{4K_0 f L} \right) \left(\frac{\Delta p_c}{p_m} \right) \quad (32)$$

In accord with Equation (14), the term $\alpha_0 p_m^n$ represents the value of α at the pressure p_m. That value, α_m, of the filtration resistance is the desired quantity. Thus it is possible to substitute α_m in Equation (32) and then solve for α_m/α_{av} as follows

$$\frac{\alpha_m}{\alpha_{av}} = \frac{4K_0 f L}{D} \frac{p_m}{\Delta p_c} \left[\frac{e^{-4K_0 f(1-n)L/D} - (p_T/p_m)^{1-n}}{1 - e^{4K_0 f(1-n)L/D}} \right] \quad (33)$$

The values of Δp_c and L/D can be fixed independently in an experiment, and α_{av} and p_T are measured. Then α_m can be calculated. Thus the term on the right-hand side of Equation (32) is a correction factor which converts the experimental value α_{av} into the desired value α_m.

If it is assumed that the hydraulic pressure drop and the weight of the cake can be neglected, then it is possible to use Equation (30) to relate p_z to the depth z. Under those circumstances, the hydraulic pressure drop is given by

$$-\Delta p_c = \frac{\gamma_s}{g} (1 - \epsilon_{av}) q \int_0^L \alpha_x dz \quad (34)$$

where the average value of the porosity is assumed to be accurate enough and g is assumed constant. Then if α_x is related to the compressive pressure through Equation (14) and the depth by Equation (30), integration of (34) gives

$$\Delta p_c = \frac{\gamma_s}{g} (1 - \epsilon_{av}) q \alpha_0 p_m^n \frac{1 - e^{-4K_0 f n L/D}}{4K_0 f n/D} \quad (35)$$

Combination with Equations (30) and (31) and rearrangement yields

$$\alpha_0 p_m^n = \alpha_m = \alpha_{av} \frac{4K_0 f n L/D}{1 - e^{-4K_0 f n L/D}} = \alpha_{av} \frac{n \ln (p_m/p_T)}{1 - (p_T/p_m)^n} \quad (36)$$

The correction factors which change α_{av} to α_m are given in Equation (36). In the case where α_x cannot be related to the compressive pressure by the simple power functions, numerical procedures must be used to obtain the correction factors.

To obtain a correction factor for the porosity, the average experimental value can be related to the point value by means of

$$\epsilon_{av} = \frac{1}{L} \int_0^L \epsilon_x d \quad (37)$$

Substitution of Equations (13) and (30) in (37) yields

$$\epsilon_{av} = \epsilon_0 p_m^{-\lambda} \frac{e^{4K_0 f \lambda L/D} - 1}{4K_0 f \lambda L/D} \quad (38)$$

$$= \epsilon_{av} \frac{(p_m/p_T)^\lambda}{(p_m/p_T)^\lambda - 1} \quad (39)$$

EXPERIMENTAL APPARATUS

The general arrangement of the C-P cell with the floating top and bottom pistons is shown in Figure 1. In Figure 4, details for the upper pistons and cell body are illustrated. The cell body illustrated in Figure 4 was made of stainless steel. Similar cells were constructed of acrylic resin and teflon lined steel. The pistons were made of mild steel and were given a 0.008 in. hard chromium plate and then ground to a diameter of 2.022-2.023 in. Other size cells of a similar nature were constructed.

The top load was applied by means of levered weights through a ball contact. A strain gauge transducer was attached to the bottom piston to measure the transmitted load. For very low pressures the top piston was counterbalanced by weights. If any portion of the weight of the upper piston is carried by the pressure of the fluid through buoyancy effects, corrections must be made. Similarly any liquid which is contained in the bottom piston must be accounted for in the transmitted load.

Formation of a representative cake can be very difficult to accomplish. For particles which are highly irregular, the method of deposition can affect the cake structure. Where there is a wide range of particle sizes, further complications develop particularly when the cake is formed by a settling process. Even if all the problems of irregular particles of differing sizes are solved, it is very difficult to lay down a cake at very low compressive pressures. In filtration, the solid particles in the surface layer of a cake are under virtually zero stress. To duplicate the surface conditions of a cake being formed in a filtration process, it is necessary to deposit the cake in a C-P cell under as close

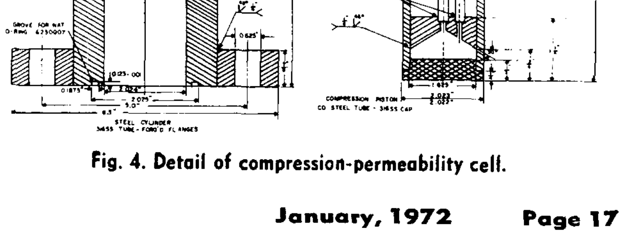

Fig. 4. Detail of compression-permeability cell.

to zero compressive pressure as possible. The cake may be formed by a deposition from a slurry or by adding liquid to the dry solid and then spooning the wet cake into the cell. Both methods were employed, depending upon the information desired. It takes hours to form a thick cake of a material with a very low permeability when low heads must be used.

In order to deposit uniform thick cakes at low pressures, a special procedure was developed. A caking piston was designed for formation of the solid bed in the C-P cell. The caking piston permitted a slurry to be circulated from a reservoir through a Sigmoid pump to the caking piston and then back to the reservoir. The bottom of the caking piston contained a large number of 1/16 in. holes through which the slurry could pass. Agitation was maintained above the perforated face of the piston by recirculation. Below the face of the piston, the slurry was quiescent; and the cake was formed by settling. The caking piston was kept slightly above the surface of the cake. A linear plot of mass of dry solids deposited against cake height was developed in order to test for uniformity. There are many sources for introducing errors; and the experimenter must be continually alert if he expects to obtain useful data.

EXPERIMENTAL RESULTS

Methodology employed in C-P cell testing can have a profound effect on results. It is generally assumed in filtration theory that the local porosity and permeability are functions of the compressive pressure alone and not time. Normally it takes a finite time for an apparent equilibrium to be established. With both soils and some filter beds, secondary creep effects may last for days or even weeks. When a load is first applied to a bed in a C-P cell, the

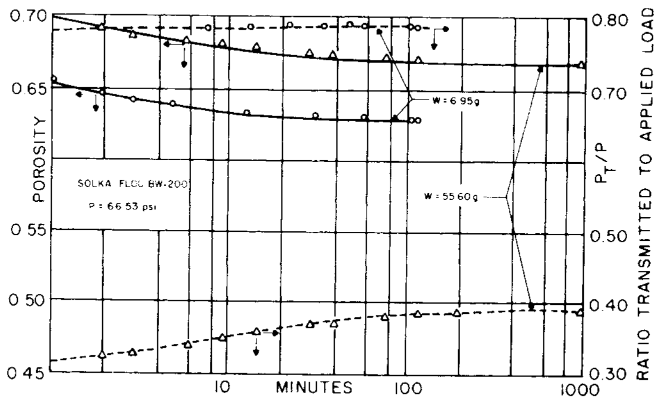

Fig. 5. Average porosity as a function of time.

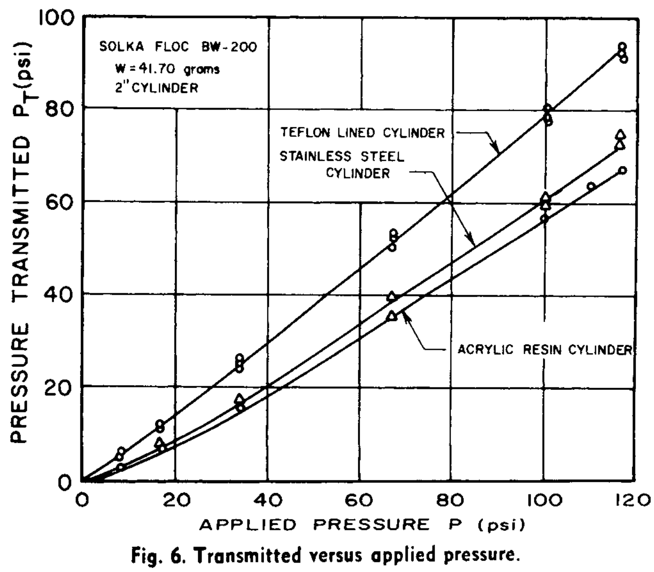

Fig. 6. Transmitted versus applied pressure.

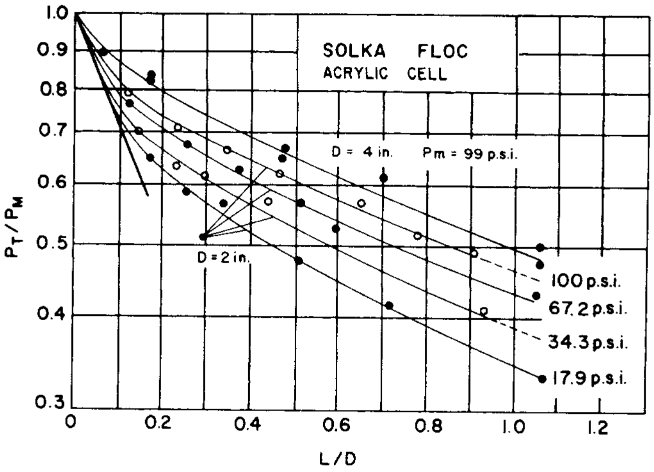

Fig. 7. Fraction of applied pressure as a function of *L/D*.

pressure is carried partially by the liquid (neutral pressure) and partly by the grains forming the cake. As the liquid is squeezed out, the neutral pressure drops to zero, the time depending upon the permeability and size of the sample. Once the entire compressive force is carried by the particles, further rearrangement may continue, particularly for very small particles with high initial porosity. Coarse sand, gravel, and materials such as tower packing reach stable conditions with low porosities (30-40%) in a very short period of time.

In Figure 5, the effect of time on the porosity and transmitted load is shown for two different thicknesses of Solka Floc BW-200. The permeability is very high, and the neutral pressure would be expected to drop to zero quickly. Thus the relatively slow approach to equilibrium would be primarily due to particle rearrangement. The cake thicknesses shown on the graph are the initial values. As the porosity decreases, the cake thickness lessens. It is impossible to maintain constant L/D ratios because of the continually changing cake thickness.

In Figure 6, the effect of different materials of construction is illustrated. As expected, the teflon coated cylinder exhibited less friction than the other materials. The theory as represented by Equation (30) predicts linearity between the applied and the transmitted load. As seen in Figure 6, the teflon coating approaches linearity more than the other materials. At an applied pressure of 100 lb./sq. in., the acrylic and stainless steel cylinders transmit about 60% of the load while the teflon transmits 80%.

In Figure 7, the logarithm of the ratio p_T/p is plotted against L/D with solka floc in an acrylic cell. According to the theoretical analysis, all of the data should fall on a single straight line. It is apparent that the basic assumptions are not valid over a wide range of L/D. Under $L/D = 0.1$, the curves tend to come together for the 2-in. cylinder. The curve for the 4-in. cell lies above the 2 in. curves and is somewhat straighter. From this graph, it would be concluded that the theory would only be applicable to very thin cakes. Further, it appears that the diameter itself influences the results as well as the ratio L/D. In the next paper in this series, further analysis will indicate that the assumption of a uniform compressive pressure at a given depth of cake in the cell is invalid for small cells and for large values of L/D.

Grace (*1*) suggested that good reproducibility could be obtained by restricting the L/D ratio to less than 0.6. However, inspection of Figure 7 indicates that only about 50% of the pressure would be transmitted to the bottom of the cell for the conditions involved. For each pair of substances forming the cake and the cell wall, a different value of $K_0 f$ would be involved. Transmitted pressure would be higher for walls made of teflon. If it were desired to keep the ratio of transmitted to applied pressure above 0.9, an L/D ratio of 0.05 or less would be required. As cakes of less than 0.25 in. are somewhat difficult to prepare, a 5-in. cell would be required if L/D were maintained at 0.05. In general, the data point to the desirability of constructing much larger cells than have been previously reported in the literature (*1, 3, 5, 7, 10, 16*).

In addition to difficulties with small diameter C-P cells, it would not be expected that similar small filters would give reliable data capable of extrapolation to plant scale.

If the low L/D range is used in Figure 7, the value of K_0f comes out to be in the range of 0.8 which is quite large in comparison to values of K_0 and f normally quoted in the soil mechanics field. If the slope of the curves at the higher L/D ratios is employed, the value of K_0f is in the range of 0.1 to 0.2. The value of K_0 would be unity for a material in which the pressure was the same in all directions. Under some conditions, certain clays can give a value of K_0 of about unity. However, most values of K_0 would be expected to be smaller and in the range of 0.5 or less. The value of f must be less than unity. It is apparent that a value of K_0f as high as 0.8 would be suspect. Sawamoto (*6*) calculated values of K_0f ranging from about 0.25 to 2.5 with a system consisting of chosen kaolin and a gun metal cell. Shirato et al. (*8*) introduced the concept of cohesion between the solids and the wall in order to modify the theory and account for the abnormally high values of K_0f. It does not appear that any reasonable value of cohesion would materially change the plots given in Figure 7.

Frequently in speaking of wall effect, the ratio of the diameter of the particles (or tower packing) to the container diameter is utilized. The particle diameters in C-P cells have magnitudes normally measured in microns (10^{-6} meter), and it is apparent that the ratio of particle to cell diameter is exceedingly small. That ratio would basically control the effect of the wall on the flow patterns in the solid itself. With such small particles, the wall would not be expected to exert any appreciable effect if the solids were uniformly packed. However, the wall friction distorts the uniformity in the C-P cell and affects the flow relations in a different manner.

There is substantial friction even with large particles such as coarse sand, gravel, and packing materials. However, after vibration, such materials frequently do not change their configurations when subjected to stresses. Thus a uniform bed may be prepared even if there is substantial wall friction. The stress pattern introduced by wall friction profoundly affect substances which change porosity with compressive pressure. Conversely, if the porosity is unaffected by compressive stresses [$\lambda = 0$ in Equation (12)], then the stress patterns will not affect the flow through the porous media.

CONCLUSION

Considerable difficulty is encountered in preparing uniform compressible cakes in small diameter cells. Wall friction profoundly affects the uniformity of packing of materials normally encountered in filtration. In C-P cells or small laboratory filters, it is important to maintain a very low L/D ratio, and to utilize large diameter cells. Data presented in this paper point to the desirability of employing cells with diameters of at least 6 to 8 in.

An investigator who plans to pass a fluid through a porous media should carefully evaluate the effect of wall friction and compressive stress on the uniformity of packing.

ACKNOWLEDGMENT

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NOTATION

A = cross-sectional area, sq. ft.
B = quantity defined in Equation (26), lb.$_f$/(sq. ft.)(ft.)
D = diameter, ft.
f = frictional coefficient, dimensionless
F_s = drag produced by liquid flow, lb.$_f$
F_w = wall friction, lb.$_f$
g = acceleration of gravity, ft./sec.^{2}
g_c = conversion factor, lb./lb.$_f$ or (lb.$_m$)(ft.)/lb.$_f$)(sec.^{2})
K_0 = ratio of lateral to vertical pressure, dimensionless
L = cake thickness, ft.
n = compressibility coefficient defined by Equation (14), dimensions meaningless
p = applied hydraulic pressure in filtration, lb.$_f$/sq. ft.
p_B = pressure at bottom of cake surface, lb.$_f$/sq. ft.
p_H = total local hydraulic pressure, lb.$_f$/sq. ft.
p_i = pressure below which α_x and ϵ_x are considered constant, lb.$_f$/sq. ft.
p_m = applied mechanical pressure, lb.$_f$/sq. ft.
p_0 = hydraulic pressure at top surface of cake, lb.$_f$/sq. ft.
p_s = solid compressive pressure, lb.$_f$/sq. ft.
p_T = transmitted pressure, lb.$_f$/sq. ft.
p_w = wall friction expressed as F_w/A, lb.$_f$/sq. ft.
p_x = hydraulic pressure at distance x in a filter cake or depth z in a C-P cell, lb.$_f$/sq. ft.
p_z = total compressive pressure at depth z in C-P cell, lb.$_f$/sq. ft.
Δp_c = total pressure drop across filter cake or C-P cell, lb.$_f$/sq. ft.
q = superficial flow rate, cu. ft./(sq. ft.) (sec.)
z = depth measured from top of cake, ft.

Greek Letters

α_i = value of α when $p_s < p_i$, ft./lb.$_m$
α_m = desired value of α in C-P cell when pressure equals p_m, ft./lb.$_m$
α_0 = constant in Equation (14), dimensions meaningless
α_R = average value of α in filtration as defined by Equation (1), ft./lb.$_m$
α_{av} = average value as found experimentally in C-P cell, ft./lb.$_m$
γ = specific weight of liquid, lb.$_f$/cu. ft.
γ_s = true specific weight of solid, lb.$_f$/cu. ft.
γ_x = $\gamma\epsilon_x + \gamma_s(1 - \epsilon_x)$, lb.$_f$/cu. ft.
γ_{av} = average value of γ_x over entire C-P cell, lb.$_f$/cu. ft.
γ_{avz} = average value of γ_x over first z feet, lb.$_f$/cu. ft.
ϵ = porosity, volume fraction of voids, dimensionless
ϵ_i = porosity when $p_s < p_i$, dimensionless
ϵ_m = desired value of ϵ in C-P cell when pressure equals p_m, dimensionless
ϵ_0 = constant in Equation (12), dimensions meaningless
ϵ_x = local porosity at depth z, dimensionless
ϵ_{av} = average value of ϵ_x over entire C-P cell, dimensionless
ϵ_{avz} = average value of ϵ_x over first z ft., dimensionless

λ = exponent in Equation (12), dimensions meaningless
μ = viscosity, $lb._m/(ft.)(sec.)$
ρ = density of liquid, $lb._m/cu.\ ft.$
ρ_s = true density of solid, $lb._m/cu.\ ft.$
ρ_x = $\rho\epsilon_x + \rho_s(1 - \epsilon_x)$, $lb._m/cu.\ ft.$
σ_r = stress in radial direction, $lb._f/cu.\ ft.$
σ_z = stress in vertical direction, same as p_z, $lb._f/sq.\ ft.$
τ_{rz} = frictional shearing force at wall, $lb._f/sq.\ ft.$
ψ_x = pressure, $lb._f/sq.\ in.$

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Process Optimization by the "Complex" Method

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The "complex" method of M. J. Box has been adapted and used to optimize the design of continuous chemical processes. Such a design involves the extremization of a nonlinear objective function subject to nonlinear equality and inequality constraints. The method consists of finding an original feasible "complex" of solutions, eliminating the worst of these by reflection through the centroid of those remaining, and repeating until an optimum has been reached. An example of significant complexity has been solved and the results are reported. The method looks quite promising for use in the optimization of chemical process designs.

Most problems in chemical engineering design and plant operation have at least several, and possibly an infinite, number of solutions. Selecting the "best" answer to such a problem out of the multiplicity of possible solutions is certainly not a new concept to chemical engineers, but it is rapidly becoming an extremely important part of chemical engineering practice. With the advent of high speed digital computers and sophisticated mathematical techniques for the calculation of optimum conditions, it is now important not only to optimize the objective function, but also to optimize the optimization procedure. In other words, it is desired to find the optimum conditions in the most efficient manner. For this optimization to be done analytically, a mathematical model of the system to be optimized must be known, including the objective function to be extremized plus all of the constraints on the system.

The nonlinear optimization of a complex chemical plant design can be represented mathematically as follows

Maximize

$$F(X_1, X_2, \ldots, X_n) \tag{1}$$

Subject to

$$G_i(X_1, X_2, \ldots, X_k) = 0 \tag{2}$$

$$i = 1, 2, \ldots, m$$

$$L_j \leq X_j \leq U_j \tag{3}$$

$$j = 1, 2, \ldots, k$$

$F(X_1, X_2, \ldots, X_n)$ is the objective function to be extremized, and is in general a nonlinear function of the n independent variables. $G_i(X_1, X_2, \ldots, X_k)$ are the m equality

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